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Bounds on the Mass for the High Dimensional Gaussian Lattice Field between Two Hard Walls

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Abstract We consider the $d(\geq 3)$-dimensional Gaussian lattice field confined between two hard walls. We show that the field becomes massive and identify the precise asymptotic behavior of the mass and the variance of the field as the height of the wall goes to infinity.

Keywords Gaussian field · Hard wall · Random interface · Mass · Random walk representation

1 Introduction and Main Results

Let $d \geq 3$, $\Lambda_N = [-N, N]^d \cap \mathbb{Z}^d$. For a configuration $\phi = \{\phi_x\}_{x\in\Lambda_N} \in \mathbb{R}^{\Lambda_N}$, consider the following massless Hamiltonian with quadratic interaction potential:

$$H_N(\phi) = \frac{1}{8d} \sum_{\substack{\{x,y\}\cap\Lambda_N \neq \phi \\ |x-y|=1}} (\phi_x - \phi_y)^2.$$

The corresponding Gibbs measure with 0-boundary conditions is defined by

$$P_N(d\phi) = \frac{1}{Z_N} \exp\{-H_N(\phi)\} \prod_{x\in\Lambda_N} d\phi_x \prod_{x\notin\Lambda_N} \delta_0(d\phi_x),$$

where $d\phi_x$ denotes Lebesgue measure on $\mathbb{R}$ and Z_N is a normalization factor. By summation by parts, this coincides with the law of the centered Gaussian lattice field on $\mathbb{R}^{\Lambda_N}$ whose covariance matrix is given by the inverse of a discrete Laplacian on Λ_N with Dirichlet boundary conditions outside Λ_N. The configuration ϕ is interpreted as an effective modelization of a random phase separating interface embedded in $d+1$-dimensional space and the spin

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ϕ_x at site $x \in \Lambda_N$ denotes its height. This model is called a lattice free field or a harmonic crystal. One of the important property of this model is that the field has the following long range correlation:

$$E^{P_\infty}[\phi_0\phi_x] \sim \frac{C}{|x|^{d-2}} \quad \text{as } |x| \to \infty,$$

where P_∞ denotes the thermodynamic limit of P_N, namely the law of the centered Gaussian field on $\mathbb{R}^{\mathbb{Z}^d}$ with covariance matrix $(-\Delta)^{-1}$, Δ is a discrete Laplacian on $\mathbb{Z}^d$. Because of this strong correlation the field exhibits many interesting behaviors, especially under the effect of various external potentials (wall, pinning, etc.) and its study has been quite active in recent years (cf. [9, 15] and references therein).

In this paper, we discuss the property of the field under the condition that the field lies between two hard walls. The corresponding event is given by

$$\mathcal{W}_N(L) = \{\phi : |\phi_x| \le L \quad \text{for every } x \in \Lambda_N\}.$$

This problem was originally investigated by Bricmont et al. [5] as a simplified model of commensurate/incommensurate transitions. They showed that under the two walls condition the field becomes massive and the following large L asymptotics holds if $d \ge 3$:

- Probability (free energy):

$$-\frac{1}{N^d}\log P_N(\mathcal{W}_N(L)) = e^{-O(L^2)}, \tag{1.1}$$

  for every $N \ge N_0(L)$.
- Mass (inverse correlation length):

$$\lim_{|x|\to\infty} -\frac{1}{|x|}\log E^{P^L_\infty}[\phi_0\phi_x] = e^{-O(L^2)}. \tag{1.2}$$

- Variance:

$$0 \le \mathrm{Var}_{P_\infty}(\phi_0) - \mathrm{Var}_{P^L_\infty}(\phi_0) \le e^{-O(L^2)}, \tag{1.3}$$

where P^L_∞ denotes the thermodynamic limit of the conditioned measure $P_N(\,\cdot\,|\mathcal{W}_N(L))$. The notation $O(k)$ denotes a function such that $c_-k \le O(k) \le c_+k$ for all large k and some constants $0 < c_- \le c_+ < \infty$ independent of k.

Remark 1.1 Actually, [5] also studied the lower dimensional cases $d = 1, 2$ and asymptotic behavior of quantities above are different in these cases. See Remark 2.3 below. [15, Sect. 4] gives a nice review about this topic.

In the previous paper [13], we gave the precise asymptotics of the free energy and the right hand side of (1.1) is $e^{-\frac{1}{2G}L^2(1+o(1))}$, where $G = (-\Delta)^{-1}(0,0)$. The path behavior under $P_N(\,\cdot\,|\mathcal{W}_N(L))$ was also studied in that paper. The main aim here is to make refinement of mass (1.2) and variance (1.3). For mathematical rigorousness of the proof, we treat the following slightly modified model: let T_N be a d-dimensional lattice torus with size $2N$ (we identify N and $-N$ in Λ_N) and consider the following Hamiltonian with quadratic interaction potential and self-potential:

$$H_{N,m}(\phi) = \frac{1}{8d}\sum_{\substack{\{x,y\}\subset T_N\\ |x-y|=1}} (\phi_x - \phi_y)^2 + \frac{1}{2}m^2\sum_{x\in T_N}\phi_x^2.$$

$P_{N,m}$ is the corresponding Gibbs measure on $\mathbb{R}^{T_N}$ with periodic boundary conditions and $P^L_{\infty,m}$ denotes the thermodynamic limit of the conditioned measure $P_{N,m}(\cdot\,|\mathcal{W}_{T_N}(L))$ where $\mathcal{W}_{T_N}(L)=\{\phi:|\phi_x|\le L \quad \text{for every } x\in T_N\}$. Note that the family of conditioned measures automatically satisfies tightness.

The reason why we consider this modified model is that we would like to use "chessboard estimate" for the proof of the lower bound of mass. For such an estimate, we need "reflection positivity" of the field and this property holds only under periodic boundary conditions (cf. Sect. 2 of [8]. See also a recent review [1] about this technique). However, since our Hamiltonian enjoys a continuous symmetry: $H_N(\phi)=H_N(\phi+c)$ for every $c\in\mathbb{R}$, the corresponding Gibbs measure is not well-defined under the periodic boundary conditions. Therefore we put mass term $\frac{1}{2}m^2\sum_{x\in T_N}\phi_x^2$ to our original Hamiltonian $H_N(\phi)$ and defined the Gibbs measure corresponding to $H_{N,m}(\phi)$ with periodic boundary conditions.

Now we are in the position to state the main result of this paper:

Theorem 1.1 *Let $d\ge 3$. For every $\delta>0$ there exists $L_0=L_0(\delta)>0$ such that for every $L\ge L_0$ it holds that*

$$\liminf_{m\to 0}\liminf_{k\to\infty}\left\{-\frac{1}{k}\log E^{P^L_{\infty,m}}[\phi_0\phi_{[kz]}]\right\}\ge e^{-(\frac{1}{4G}+\delta)L^2}, \tag{1.4}$$

$$\limsup_{m\to 0}\limsup_{k\to\infty}\left\{-\frac{1}{k}\log E^{P^L_{\infty,m}}[\phi_0\phi_{[kz]}]\right\}\le e^{-(\frac{1}{4G}-\delta)L^2}, \tag{1.5}$$

for every $z\in\mathbb{S}^{d-1}=\{z\in\mathbb{R}^d;|z|=1\}$ and

$$\liminf_{m\to 0}\{\mathrm{Var}_{P_\infty}(\phi_0)-\mathrm{Var}_{P^L_{\infty,m}}(\phi_0)\}\ge e^{-(\frac{1}{2G}+\delta)L^2}, \tag{1.6}$$

where $[kz]$ is the integral part of kz, componentwise. Also, if $d\ge 5$ then we have

$$\limsup_{m\to 0}\{\mathrm{Var}_{P_\infty}(\phi_0)-\mathrm{Var}_{P^L_{\infty,m}}(\phi_0)\}\le e^{-(\frac{1}{2G}-\delta)L^2}. \tag{1.7}$$

We stress that in this result the mass term of the Hamiltonian serves only to replace the 0-boundary conditions and to make the model well-defined. It is well-known that once we put the self-potential $\frac{1}{2}m^2\sum_{x\in T_N}\phi_x^2$, the field becomes massive and the mass vanishes as $m\to 0$:

$$\lim_{k\to\infty}\left\{-\frac{1}{k}\log E^{P_{\infty,m}}[\phi_0\phi_{[kz]}]\right\}=m(1+o(1)),$$

for every $z\in\mathbb{S}^{d-1}=\{z\in\mathbb{R}^d;|z|=1\}$ where $P_{\infty,m}$ denotes the thermodynamic limit of $P_{N,m}$, the law of the centered Gaussian field on $\mathbb{R}^{\mathbb{Z}^d}$ with covariance matrix $(m^2-\Delta)^{-1}$. Theorem 1.1 means that by the confinement effect, the mass does not vanish even though we take the limit $m\to 0$ and this implies that the massless field becomes massive by two hard walls. The result also identifies precise large L asymptotics of the mass and variance. The critical behavior of this type for massless Gaussian field was studied by [2] and [4] in the case with δ-pinning instead of two walls. [4] identified the precise asymptotics of mass and variance as the strength of pinning goes to 0.

Next, we explain our strategy of the proof. The proof of the lower bound of mass is based on a combination of random walk representation of correlation of the field by Brydges-Fröhlich-Spencer [6] and reflection positivity, in the form of chessboard estimate. Actually this strategy is similar to that of [6] and they showed the exponential decay of correlations with rough estimate of mass. This type of argument was also used for the δ-pinning case by [2]. The ingredient of our proof is that we know the precise asymptotic behavior of the free energy. By combining the strategy of [6] with an estimate derived from the asymptotics of the free energy, we can obtain the precise lower bound of the mass. The proof of the upper bound is also based on the random walk representation. Especially, an estimate on the number of multiple points of pinned random walk plays an important role in the argument. Variance estimate is proved by a modified argument to the proof of mass estimate. We remark that our proof fairly differs from that of [5] which is based on the technique of field theory and some of their arguments have difficulty to follow.

Finally, we give some remarks about the result.

Remark 1.2 Since our model (random walk representation) does not have suitable subadditive property, unfortunately we have not obtained the existence of the limit $\lim_{k\to\infty}\{-\frac{1}{k}\log E^{P^L_{\infty,m}}[\phi_0\phi_{[kz]}]\}$.

Remark 1.3 Random walk representation of [6] holds only for the Gaussian setting. Hence we cannot apply our proof to the non-Gaussian case. With regard to our problem in the non-Gaussian setting, only probability estimate of $\mathcal{W}_N(L)$ is known by [14] in the context of massless fields with strictly convex interaction potentials.

The rest of this paper is organized as follows. We give the proof of the mass lower bound (1.4) and upper bound (1.5) in Sects. 2 and 3, respectively. The proof of variance estimates (1.6) and (1.7) is given in Sect. 4. We remark that throughout this paper below, C represents a positive constant which does not depend on the size of the system N, height of the wall L and mass m but may depend on other parameters. Also, this C in estimates may change from place to place in the paper.

2 Lower Bound of Mass

In this section, we prove (1.4). At first we recall Brydges-Fröhlich-Spencer's random walk representation (cf. [6, Theorem 2.2]) which is applied to our setting.

Lemma 2.1 *It holds that*

$$E^{P_{N,m}}[\phi_0\phi_x|\mathcal{W}_{T_N}(L)] = \sum_{\omega:0\to x}{}' \left(\frac{1}{2d(m^2+1)}\right)^{|\omega|}\int \frac{\Xi^L_{N,m}(\frac{2}{m^2+1}\psi)}{\Xi^L_{N,m}}\mu_\omega(d\psi),$$

where the primed sum represents a summation with respect to paths of simple random walk on T_N connecting 0 and x. For a path ω, $|\omega|$ denotes its length. $\mu_\omega(d\psi) = \prod_{z\in T_N}\mu_{n(z,\omega)}(d\psi_z)$ is a product measure, μ_n is a measure on $[0,\infty)$ defined by

$$\mu_n(dt) = \begin{cases} \delta_0(dt) & \textit{if } n=0,\\ e^{-t}\frac{t^{n-1}}{(n-1)!}I(t\geq 0)dt & \textit{if } n\in\mathbb{N}. \end{cases}$$

$n(z,\omega)$ denotes the total number of visits of site $z \in T_N$ in the path ω. Also,

$$\Xi^L_{N,m}\left(\frac{2}{m^2+1}\psi\right) = P_{N,m}\left(\phi_z^2 + \frac{2}{m^2+1}\psi_z \le L^2 \textit{ for every } z \in T_N\right),$$
$$\Xi^L_{N,m} = \Xi^L_{N,m}(\underline{0}) = P_{N,m}(\mathcal{W}_{T_N}(L)).$$

Since the interaction of our Hamiltonian determines from the difference of the height of the neighboring sites and we consider the model with periodic boundary conditions, reflection positivity holds with respect to all reflections which are parallel to each axis. Hence we can use chessboard estimate and we have

$$\Xi^L_{N,m}\left(\frac{2}{m^2+1}\psi\right) \le \prod_{y\in T_N} P_{N,m}\left(\phi_z^2 + \frac{2}{m^2+1}\psi_y \le L^2 \text{ for every } z \in T_N\right)^{\frac{1}{|T_N|}}.$$

Note that the right hand side is a product of the probability of confinement between two walls and the height of the wall is spatially homogeneous. Then,

$$\begin{aligned}
I_\omega &\equiv \int \frac{\Xi^L_{N,m}(\frac{2}{m^2+1}\psi)}{\Xi^L_{N,m}} \mu_\omega(d\psi) \\
&\le \int \prod_{y\in T_N} \left(\frac{P_{N,m}(\phi_z^2 + \frac{2}{m^2+1}\psi_y \le L^2 \text{ for every } z \in T_N)}{P_{N,m}(\mathcal{W}_{T_N}(L))}\right)^{\frac{1}{|T_N|}} \mu_\omega(d\psi) \\
&= \prod_{y\in T_N} \int P_{N,m}\left(\mathcal{W}_{T_N}\left(\left(L^2 - \frac{2}{m^2+1}\psi_y\right)^{\frac{1}{2}}\right) \,\middle|\, \mathcal{W}_{T_N}(L)\right)^{\frac{1}{|T_N|}} \mu_{n(y,\omega)}(d\psi_y) \\
&\equiv \prod_{y\in T_N} q^L_{N,m}(n(y,\omega)).
\end{aligned}$$

Now $q^L_{N,m}(0) = 1$ and $q^L_{N,m}(n)$ is decreasing in n (cf. the proof of Theorem 4.4 of [6]). Hence we obtain

$$I_\omega \le \prod_{\substack{y\in T_N \\ n(y,\omega)\ge 1}} q^L_{N,m}(1) = (q^L_{N,m})^{|R(\omega)|},$$

where $q^L_{N,m} = q^L_{N,m}(1)$ and $R(\omega) = \{z \in T_N; n(z,\omega) \ge 1\}$ is the range of the random walk path ω. Combining this estimate with Lemma 2.1,

$$\begin{aligned}
E^{P_{N,m}}[\phi_0\phi_x \mid \mathcal{W}_{T_N}(L)] &\le {\sum_{\omega:0\to x}}' \left(\frac{1}{2d(m^2+1)}\right)^{|\omega|} (q^L_{N,m})^{|R(\omega)|} \\
&= \sum_{k=0}^\infty \left(\frac{1}{2d(m^2+1)}\right)^k {\sum_{\substack{\omega:0\to x \\ |\omega|=k}}}' (q^L_{N,m})^{|R(\omega)|}. \qquad (2.1)
\end{aligned}$$

We have the following estimate for $q^L_{N,m}$. The proof is given later.

Lemma 2.2 *There exists a constant* $C_1 > 0$ *such that for every* $\delta > 0$, *we have*

$$\limsup_{N\to\infty} q^L_{N,m} \le 1 - C_1 e^{-(\frac{1}{2G_m}+\delta)L^2},$$

for every L *large enough and every* $m > 0$, *where* $G_m = (m^2 - \Delta)^{-1}(0,0)$.

By this lemma and (2.1), taking the limit $N \to \infty$ (if necessary along the subsequence), we obtain

$$\begin{aligned} E^{P^L_{\infty,m}}[\phi_0\phi_x] &\le \sum_{k=0}^{\infty}\left(\frac{1}{m^2+1}\right)^k \mathbb{E}_0[I(\eta_k = x)(1-p^L_m)^{|\eta[0,k]|}] \\ &\le \sum_{k=0}^{\infty}\sum_{l=0}^{k} \mathbb{E}_0[I(T_x = l)I(\eta_k = x)(1-p^L_m)^{|\eta[0,T_x]|}] \\ &= \sum_{l=0}^{\infty}\sum_{k=l}^{\infty} \mathbb{E}_0[I(T_x = l)(1-p^L_m)^{|\eta[0,T_x]|}]\mathbb{P}_0(\eta_{k-l} = 0) \\ &= G\mathbb{E}_0[(1-p^L_m)^{|\eta[0,T_x]|}], \end{aligned}$$

where $p^L_m = C_1 e^{-(\frac{1}{2G_m}+\delta)L^2}$, $\{\eta_k\}_{k\ge 0}$ is a simple random walk on $\mathbb{Z}^d$ and $\eta_{[0,k]}$ denotes the set of points visited by the walk up to time k. $T_x = \inf\{k \ge 0; \eta_k = x\}$ is the first hitting time of site $x \in \mathbb{Z}^d$. Now, by the proof of Theorem 2.3 of [4] we know that

$$\mathbb{E}_0[(1-p^L_m)^{|\eta[0,T_x]|}] \le e^{-C(p^L_m)^{\frac{1}{2}}|x|},$$

for every $x \in \mathbb{Z}^d$ and L large enough. Combining these estimates with $\lim_{m\to 0} G_m = G$, we obtain (1.4).

The rest is to prove Lemma 2.2. For this purpose we prepare the following lemma.

Lemma 2.3 *Let* $d \ge 3$, $m > 0$ *and* $0 < \lambda < 1$. *For every* $\delta > 0$, *there exist* $L_0 = L_0(\lambda,\delta) > 0$ *large enough such that the following holds*: *for every* $L \ge L_0$, *there exists* $N_0 = N_0(L)$ *and it holds that*

$$P_{N,m}(\mathcal{W}_{T_N}(\lambda L) \mid \mathcal{W}_{T_N}(L)) \le e^{-N^d e^{-(\frac{1}{2G_m}+\delta)\lambda^2 L^2}},$$

for every $N \ge N_0$.

Proof We have only to prove for small $\delta > 0$ and take $\delta > 0$ which satisfies $(\frac{1}{2G_m}+\delta)\lambda^2 < \frac{1}{2G_m}$. At first, by Griffith's inequality and Markov property of the field, we can divide T_N into boxes with side-length $2M$ by 0-boundary conditions and we have

$$\begin{aligned} P_{N,m}(\mathcal{W}_{T_N}(\lambda L)) &\le P^0_{M,m}(\mathcal{W}_{\Lambda_M}(\lambda L))^{(\frac{N}{M})^d} \\ &\le P^0_{M,m}(\phi_x \ge -\lambda L \text{ for every } x \in \Lambda_M)^{(\frac{N}{M})^d}, \end{aligned}$$

where $P^0_{M,m}$ denotes the law of massive Gaussian field on $\Lambda_M = [-M,M]^d \cap \mathbb{Z}^d$ with mass m and 0-boundary conditions outside Λ_M. For simplicity we assume that N is divided by M.

By following the argument of [10, Sect. 3.6], for every $\alpha > 0$ and $\delta > 0$, we can prove that

$$P^0_{M,m}(\phi_x \geq -\sqrt{\alpha \log M} \text{ for every } x \in \Lambda_M) \leq \exp\{-CM^{d-\frac{\alpha}{2G_m}-\alpha\delta}\}, \tag{2.2}$$

for every $M > 0$ large enough, where $C > 0$ is a constant independent of M and m. We choose now $M = e^{\frac{1}{\alpha}\lambda^2 L^2}$. Then, $\lambda L = \sqrt{\alpha \log M}$ and we have

$$\begin{aligned} P^0_{M,m}(\mathcal{W}_{\Lambda_M}(\lambda L)) &\leq \exp\{-CM^{d-\frac{\alpha}{2G_m}-\alpha\delta}\}^{(\frac{N}{M})^d} \\ &\leq \exp\{-CN^d e^{-(\frac{1}{2G_m}+\delta)\lambda^2 L^2}\}, \end{aligned}$$

for L, N large enough (N depends on L).

On the other hand, by Griffith's inequality and Gaussian tail estimate, we have

$$\begin{aligned} P_{N,m}(\mathcal{W}_{T_N}(L)) &\geq \prod_{x \in T_N} P_{N,m}(|\phi_x| \leq L) \\ &\geq \prod_{x \in T_N} \left[1 - \exp\left\{-\frac{L^2}{2\text{Var}_{P_{N,m}}(\phi_x)}\right\}\right] \\ &\geq \exp\{-CN^d e^{-\frac{L^2}{2G_m}}\}. \end{aligned}$$

Collecting all the estimates, we obtain

$$\begin{aligned} P_{N,m}(\mathcal{W}_{T_N}(\lambda L)|\mathcal{W}_{T_N}(L)) &\leq \exp\{-CN^d e^{-(\frac{1}{2G_m}+\delta)\lambda^2 L^2} + CN^d e^{-\frac{L^2}{2G_m}}\} \\ &\leq \exp\{-CN^d e^{-(\frac{1}{2G_m}+\delta)\lambda^2 L^2}\}, \end{aligned}$$

for L and N large enough (N depends on L). □

Remark 2.1 The proof of the estimate (2.2) is non-trivial and this is given along the proof of entropic repulsion for massless field. Since we consider massive Gaussian field, hypercontractivity (cf. [3, Proposition A.18]) would be a powerful tool. However, if we apply hypercontractivity, we obtain the estimate (2.2) with the constant $C = C(m)$ depending on the mass m and this constant goes to 0 as $m \to 0$. Therefore hypercontractivity does not help the proof of (2.2).

Proof of Lemma 2.2 Take $0 < \varepsilon < \frac{1}{2}$ and $0 < \gamma < 1 - \varepsilon$. Then,

$$\begin{aligned} q^L_{N,m} &\equiv \int P_{N,m}\left(\phi_x^2 + \frac{2}{m^2+1}t \leq L^2 \text{ for every } x \in T_N \,\Big|\, \mathcal{W}_{T_N}(L)\right)^{\frac{1}{|T_N|}} \mu_1(dt) \\ &= \int_0^1 \frac{1}{2}(m^2+1)L^2 e^{-\frac{1}{2}(m^2+1)L^2 t} \\ &\quad \times P_{N,m}(\phi_x^2 \leq L^2(1-t) \text{ for every } x \in T_N|\mathcal{W}_{T_N}(L))^{\frac{1}{|T_N|}} dt \\ &\leq 1 - \int_\gamma^{1-\varepsilon} \frac{1}{2}(m^2+1)L^2 e^{-\frac{1}{2}(m^2+1)L^2 t} \end{aligned}$$

$$\times \{1 - P_{N,m}(\phi_x^2 \le L^2(1-t) \text{ for every } x \in T_N | \mathcal{W}_{T_N}(L))^{\frac{1}{|T_N|}}\}dt$$
$$\equiv 1 - J^L_{N,m}.$$

By Lemma 2.3, for every $\delta > 0$ there exist $L_0 = L_0(\varepsilon, \delta) > 0$ such that the following holds: for every $L \ge L_0$, there exists $N_0 = N_0(L)$ and it holds that

$$P_{N,m}(\phi_x^2 \le L^2(1-t) \text{ for every } x \in T_N | \mathcal{W}_{T_N}(L))^{\frac{1}{|T_N|}} \le e^{-e^{-(\frac{1}{2G_m}+\delta)(1-t)L^2}},$$

for every $N \ge N_0$ and every $0 < t \le 1 - \varepsilon$. Then, we have

$$J^L_{N,m} \ge \int_\gamma^{1-\varepsilon} \frac{1}{2}(m^2+1)L^2 e^{-\frac{1}{2}(m^2+1)L^2 t}\{1 - e^{-e^{-(\frac{1}{2G_m}+\delta)(1-t)L^2}}\}dt$$
$$\ge C e^{-(\frac{1}{2G_m}+\delta)L^2} \int_\gamma^{1-\varepsilon} \frac{1}{2}(m^2+1)L^2 e^{-\frac{1}{2}(m^2+1)L^2 t} e^{(\frac{1}{2G_m}+\delta)L^2 t} dt,$$

for every L, N large enough. The integral in the right most side is bounded form below by $CL^2 e^{-C\gamma L^2}$. Hence we obtain the result. □

Remark 2.2 By this proof, we can also obtain the exponential decay of the correlation:

$$\limsup_{m \to 0} E^{P^L_{\infty,m}}[\phi_x \phi_y] \le G \exp\{-e^{-(\frac{1}{4G}+\delta)L^2}|x-y|\},$$

for every $x, y \in \mathbb{Z}^d$ and L large enough.

Remark 2.3 We have a trivial estimate

$$q^L_{N,m} \le \int_0^{\frac{1}{2}(m^2+1)L^2} e^{-t} dt = 1 - e^{-\frac{1}{2}(m^2+1)L^2},$$

which holds for every $d \ge 1$. By this estimate and the argument of Sect. 4 of [2], we can prove the exponential decay of correlations and the lower bound of mass also for the case $d = 2$ with the mass $e^{-O(L^2)}$. However, this asymptotics of mass differs from the result of [5] (they stated that mass behaves as $e^{-O(L)}$ when $d = 2$).

To obtain the mass lower bound $e^{-O(L)}$ for $d = 2$, we need to show the estimate such as $\limsup_{N\to\infty} q^L_{N,m} \le 1 - e^{-O(L)}$ and this can be proved once we know the precise asymptotic behavior of $P_N(\mathcal{W}_N(L))$ (or $P_{N,m}(\mathcal{W}_N(L))$ with small $m > 0$) as $N \to \infty$ with the order $e^{-O(L)}$. But this has not been obtained yet. The current best estimate is that

$$e^{-\frac{1+\delta}{\sqrt{g}}L} \le -\frac{1}{N^d} \log P_N(\mathcal{W}_N(L)) \le e^{-\frac{1-\delta}{2\sqrt{g}}L},$$

for every $\delta > 0$ and L, N large enough (N depends on L), where $g = \frac{2}{\pi}$. This can be shown by applying the argument of [13] to the case of $d = 2$.

In the case of $d = 1$, we can show the mass lower bound by $O(L^{-2})$ instead of $e^{-O(L^2)}$. This follows from the same argument to the proof of Theorem 1.4 and Lemma 4.4 of [11].

3 Upper Bound of Mass

In this section, we prove the upper bound of mass (1.5).

Step1 At first, we estimate the summand of Lemma 2.1 from below.

$$\begin{aligned}
I_\omega &\equiv \int \frac{\Xi^L_{N,m}(\frac{2}{m^2+1}\psi)}{\Xi^L_{N,m}}\mu_\omega(d\psi) \\
&= \int P_{N,m}\left(\phi_z^2 + \frac{2}{m^2+1}\psi_z \le L^2 \text{ for every } z \in R(\omega) \,\Big|\, \mathcal{W}_{T_N}(L)\right) \\
&\quad \times \prod_{z\in R(\omega)} \mu_{n(z,\omega)}(d\psi_z) \\
&\ge \prod_{z\in R(\omega)} \int P_{N,m}\left(\phi_z^2 + \frac{2}{m^2+1}\psi_z \le L^2\right)\mu_{n(z,\omega)}(d\psi_z) \\
&= \prod_{z\in R(\omega)} P_{N,m}\otimes \mu_{n(z,\omega)}\left(\phi_z^2 + \frac{2}{m^2+1}\psi_z \le L^2\right),
\end{aligned}$$

where the first equality follows from the definition of μ_ω and the inequality follows from Griffith's inequality. Therefore we can decompose I_ω into a product of integrals of one site marginal. Combining this estimate with the random walk representation, we obtain

$$\begin{aligned}
&E^{P_{N,m}}[\phi_0\phi_x|\mathcal{W}_{T_N}(L)] \\
&\quad \ge \sum_{k\ge 0}\left(\frac{1}{2d(m^2+1)}\right)^k \sideset{}{'}\sum_{\substack{\omega:0\to x\\ |\omega|=k}} \prod_{z\in R(\omega)} P_{N,m}\otimes \mu_{n(z,\omega)}\left(\phi_z^2 + \frac{2}{m^2+1}\psi_z \le L^2\right).
\end{aligned}$$

By taking the limit $N\to\infty$,

$$\begin{aligned}
&E^{P^L_{\infty,m}}[\phi_0\phi_x] \\
&\quad \ge \sum_{k\ge 0}\left(\frac{1}{m^2+1}\right)^k \mathbb{E}_0\left[\prod_{z\in\eta_{[0,k]}} q^L_m(n(z,\eta_{[0,k]}))I(\eta_k = x)\right] \\
&\quad \ge \sum_{k\ge 0}\left(\frac{1}{m^2+1}\right)^k \mathbb{E}_0\left[\prod_{z\in\eta_{[0,k]}} q^L_m(n(z,\eta_{[0,k]})) \,\Big|\, \eta_k = x\right]\mathbb{P}_0(\eta_k = x),
\end{aligned}$$

where $q^L_m(j) = P_{\infty,m}\otimes\mu_j(\phi_0^2 + \frac{2}{m^2+1}\psi_0 \le L^2)$. Note that $P_{N,m}$ converges weakly to $P_{\infty,m}$ as $N\to\infty$ and $P_{\infty,m}$ is translation invariant. Restricting the summation to the special choice of k, to be specified later on, we get

$$\begin{aligned}
&\log E^{P^L_{\infty,m}}[\phi_0\phi_x] \\
&\quad \ge k\log\frac{1}{m^2+1} \\
&\qquad + \log\mathbb{E}_0\left[\prod_{z\in\eta_{[0,k]}} q^L_m(n(z,\eta_{[0,k]})) \,\Big|\, \eta_k = x\right] + \log\mathbb{P}_0(\eta_k = x). \qquad (3.1)
\end{aligned}$$

Step2 We estimate the second term of (3.1). By Jensen's inequality

$$\begin{aligned} J_k &\equiv \log \mathbb{E}_0\Big[\prod_{z\in\eta[0,k]} q_m^L(n(z,\eta_{[0,k]}))\;\Big|\;\eta_k = x\Big] \\ &\geq \mathbb{E}_0\Big[\sum_{z\in\eta[0,k]} \log q_m^L(n(z,\eta_{[0,k]}))\;\Big|\;\eta_k = x\Big] \\ &= \mathbb{E}_0\Big[\sum_{j\geq 1} R_k(j)\log q_m^L(j)\;\Big|\;\eta_k = x\Big] \\ &= \sum_{j\geq 1} \mathbb{E}_0[R_k(j)\mid \eta_k = x]\log q_m^L(j), \end{aligned}$$

where $R_k(j) = R_k(j,\eta_{[0,k]}) \equiv \sharp\{z\in\mathbb{Z}^d; n(z,\eta_{[0,k]}) = j\}$ denotes the total number of sites where the simple random walk visits exactly j times in the first k steps.

Now, we estimate $q_m^L(j)$ from below

$$\begin{aligned} q_m^L(j) &= P_{\infty,m}\otimes\mu_j\left(\phi_0^2 + \frac{2}{m^2+1}\psi_0 \leq L^2\right) \\ &= E^{P_{\infty,m}}\left[\mu_j\left(\psi_0 \leq \frac{1}{2}(m^2+1)(L^2-\phi_0^2)\right) I(|\phi_0|\leq L)\right]. \end{aligned}$$

Since $E^{\mu_j}[e^{\theta\psi_0}] = (\frac{1}{1-\theta})^j$ for every $0\leq\theta<1$, for given ϕ_0 with $|\phi_0|\leq L$, we have

$$\mu_j\left(\psi_0 \leq \frac{1}{2}(m^2+1)(L^2-\phi_0^2)\right) \geq 1 - e^{-\frac{1}{2}(m^2+1)(L^2-\phi_0^2)\theta}\left(\frac{1}{1-\theta}\right)^j,$$

and this yields

$$\begin{aligned} q_m^L(j) &\geq P_{\infty,m}(|\phi_0|\leq L) \\ &\quad - e^{-\frac{1}{2}(m^2+1)\theta L^2}\left(\frac{1}{1-\theta}\right)^j E^{P_{\infty,m}}[e^{\frac{1}{2}(m^2+1)\theta\phi_0^2} I(|\phi_0|\leq L)]. \end{aligned}$$

We choose now $\theta = \theta_m \equiv \frac{1}{(m^2+1)G_m}$. Then direct calculation shows

$$E^{P_{\infty,m}}[e^{\frac{1}{2}(m^2+1)\theta_m\phi_0^2} I(|\phi_0|\leq L)] = \frac{2L}{\sqrt{2\pi G_m}},$$

and combining these estimates with Gaussian tail estimate, we get

$$q_m^L(j) \geq 1 - p(L)e^{-\frac{L^2}{2G_m}}\left(\frac{1}{1-\theta_m}\right)^j,$$

where $p(L)$ represents a positive polynomial with respect to L and this may change place to place in the paper. For fixed $\delta>0$, if $j \leq [\frac{\frac{1}{2G_m}-\frac{1}{2}\delta}{-\log(1-\theta_m)}L^2] \equiv [\kappa L^2]$ then

$$e^{-\frac{L^2}{2G_m}}\left(\frac{1}{1-\theta_m}\right)^j \leq e^{-\frac{1}{2}\delta L^2},$$

and hence we obtain

$$q_m^L(j) \geq \exp\left\{-p(L)e^{-\frac{L^2}{2Gm}}\left(\frac{1}{1-\theta_m}\right)^j\right\}, \tag{3.2}$$

for L large enough in this case. On the other hand, for every $j \geq 1$ we have

$$\begin{aligned} q_m^L(j) &\geq P_{\infty,m}\left(\phi_0^2 \leq \frac{1}{2}L^2\right)\mu_j\left(\frac{2}{m^2+1}\psi_0 \leq \frac{1}{2}L^2\right) \\ &\geq (1-e^{-\frac{L^2}{4Gm}})e^{-\frac{1}{4}(m^2+1)L^2}e^{-Cj\log j} \\ &\geq e^{-CL^2}e^{-Cj\log j}, \end{aligned} \tag{3.3}$$

for L large enough, where we used a rough estimate

$$\mu_j(\psi_0 \leq a) = \int_0^a e^{-t}\frac{t^{j-1}}{(j-1)!}dt \geq e^{-a}\frac{a^j}{j!}, \quad a > 0.$$

Using estimate (3.2) for $j \leq [\kappa L^2]$ and (3.3) for $j \geq [\kappa L^2]+1$, we obtain

$$\begin{aligned} J_k \geq &\sum_{j\leq[\kappa L^2]} \mathbb{E}_0[R_k(j) \mid \eta_k = x]\left\{-p(L)e^{-\frac{L^2}{2Gm}}\left(\frac{1}{1-\theta_m}\right)^j\right\} \\ &+ \sum_{j\geq[\kappa L^2]+1} \mathbb{E}_0[R_k(j) \mid \eta_k = x]\{-p(L) - Cj\log j\} \\ \equiv\ & J_k^1 + J_k^2. \end{aligned}$$

Now we have the following lemma. The proof is given in the end of this section.

Lemma 3.1 *Let $\{\eta_n\}_{n\geq 0}$ be a simple random walk on $\mathbb{Z}^d$. If $d \geq 3$, then there exists a constant $C > 0$ such that for every $\alpha > 0$ large enough there exists $r_0(\alpha) > 0$ and it holds that for every x with $|x| \geq r_0$ and $j \geq 1$, we have*

$$\frac{1}{k}\mathbb{E}_0[R_k(j) \mid \eta_k = x] \leq C(1-\gamma)^{j-1},$$

where we set $k = [\alpha|x|]$ and $\gamma = \mathbb{P}_0(\eta_n \neq 0$ for every $n \geq 1)$. Also, if $d \geq 5$ then we have

$$\sum_{n=1}^{\infty} \mathbb{E}_0[R_n(j)I(\eta_n = x)] \leq C(1-\gamma)^{j-1},$$

for every $x \in \mathbb{Z}^d$ and $j \geq 1$.

Remark 3.1 For a simple random walk on $\mathbb{Z}^d$, $d \geq 3$, without pinned condition, it is well known that $\frac{1}{k}R_k(j) \to \gamma^2(1-\gamma)^{j-1}$ as $k \to \infty$ a.s. (cf. [7] and [12]).

For $\alpha = \alpha(L)$ which goes to ∞ as $L \to \infty$, set $k = \alpha|x|$ (we will choose α later.) Then for every L large enough and $x \in \mathbb{Z}^d$ with $|x|$ large enough ($|x|$ depends on L), we have

$$J_k^1 \geq \sum_{j \leq [\kappa L^2]} C(1-\gamma)^{j-1}\alpha|x|\left\{-p(L)e^{-\frac{L^2}{2G_m}}\left(\frac{1}{1-\theta_m}\right)^j\right\}$$

$$\geq -\alpha|x|p(L)e^{-\frac{L^2}{2G_m}} \sum_{j \leq [\kappa L^2]} \left(\frac{1-\gamma}{1-\theta_m}\right)^{j-1}$$

$$\geq -\alpha|x|p(L)e^{-\frac{L^2}{2G_m}}(\beta_m)^{[\kappa L^2]},$$

where $\beta_m = \frac{1-\gamma}{1-\theta_m}$. $\theta_m = \frac{1}{(m^2+1)G_m} \to \frac{1}{G}$ as $m \to 0$ and it is well known that $G\gamma = 1$. Therefore $\beta_m \to 1$ as $m \to 0$ and we obtain that for every $\delta > 0$ there exists $L_0 > 0$ such that for every $L \geq L_0$,

$$J_k^1 \geq -\alpha|x|e^{-(\frac{1}{2G}-\delta)L^2}, \tag{3.4}$$

for every $x \in \mathbb{Z}^d$ with $|x|$ large enough and $m > 0$ small enough ($|x|, m$ depend on L).

For J_k^2, we have

$$J_k^2 \geq \sum_{j \geq [\kappa L^2]+1} C(1-\gamma)^{j-1}\alpha|x|\{-p(L) - Cj\log j\}$$

$$\geq -\alpha|x|p(L)(1-\gamma)^{[\kappa L^2]+1}.$$

By the definition of κ, we have that $(1-\theta_m)^{[\kappa L^2]+1} \leq e^{-(\frac{1}{2G_m}-\frac{1}{2}\delta)L^2}$. Combining these estimates with the fact that $\theta_m \to \gamma$ as $m \to 0$, we obtain the same estimate as (3.4) for J_k^2.

Step3 For the third term of (3.1), we use the following lemma.

Lemma 3.2 ([4, Proposition B.2]) *There exists $a_0 > 0$ such that for $\frac{|x|}{n} < a_0$,*

$$\mathbb{P}_0(\eta_n = x) = \left(\frac{1}{(2\pi n)^{\frac{d}{2}}\sqrt{\det \mathcal{Q}(\frac{x}{n})}} + O\left(\frac{1}{n^{\frac{d+1}{2}}}\right)\right)\exp\left\{-nI\left(\frac{x}{n}\right)\right\},$$

where $\mathcal{Q}(\xi)$, $|\xi| < a_0$, are $d \times d$-matrices, depending analytically on ξ and satisfying $\mathcal{Q}(0) = \frac{1}{d}I_d$. $I(\xi)$, $|\xi| < a_0$, also depends analytically on ξ and satisfies $\nabla I(0) = 0$, $\nabla^2 I(0) = dI_d$.

Then for $k = \alpha|x|$ with $\alpha = \alpha(L)$, L large enough, we have that

$$\log \mathbb{P}_0(\eta_k = x) \geq \log\left\{\frac{C}{(\alpha|x|)^{\frac{d}{2}}}\exp\left\{-\alpha|x|I\left(\frac{x}{\alpha|x|}\right)\right\}\right\}$$

$$\geq -C\frac{|x|}{\alpha}(1+o(1)),$$

where $o(1)$ represents a term which goes to 0 as $|x| \to \infty$.

Collecting all the estimates,

$$\limsup_{m\to 0}\limsup_{k\to\infty}\left\{-\frac{1}{k}\log E^{P^L_{\infty,m}}[\phi_0\phi_{[kz]}]\right\}$$
$$\le \limsup_{m\to 0}\limsup_{k\to\infty}\left\{-\frac{1}{k}\left[\alpha k\log\frac{1}{m^2+1}+C\alpha k e^{-(\frac{1}{2G}-\delta)L^2}+C\frac{k}{\alpha}(1+o(1))\right]\right\}$$
$$\le C\left(\alpha e^{-(\frac{1}{2G}-\delta)L^2}+\frac{1}{\alpha}\right),$$

for every $z\in\mathbb{S}^{d-1}$ and L large enough. Finally by taking $\alpha=e^{\frac{1}{4G}L^2}$, we obtain (1.5). □

Remark 3.2 Since we do not need reflection positivity, this argument works well also for the case of $m\equiv 0$, the original massless Gaussian field (with 0-boundary conditions). We gave the proof of the case with periodic boundary conditions and small mass for the consistency with the lower bound.

Proof of Lemma 3.1 Let $T_x^{(0)}=0$, $T_x^{(j)}=\inf\{n>T_x^{(j-1)};\eta_n=x\}$ $(j\ge 1)$ be the j-th hitting time of site $x\in\mathbb{Z}^d$. For $j\ge 1$ and $x\in\mathbb{Z}^d$, by Markov property, we have

$$\begin{aligned}
&\mathbb{E}_0[R_n(j)I(\eta_n=x)]\\
&\quad=\sum_{y\in\mathbb{Z}^d}\mathbb{P}_0(T_y^{(j)}\le n, T_y^{(j+1)}>n,\eta_n=x)\\
&\quad=\sum_{y\in\mathbb{Z}^d}\sum_{0\le s+t\le n}\mathbb{P}_0(T_y^{(1)}=s)\mathbb{P}_0(T_0^{(j-1)}=t)q_{n-(s+t)}(x-y)\\
&\quad=\sum_{0\le t\le n}\mathbb{P}_0(T_0^{(j-1)}=t)\sum_{y\in\mathbb{Z}^d}\sum_{0\le s\le n-t}\mathbb{P}_0(T_y^{(1)}=s)q_{n-(s+t)}(x-y),
\end{aligned}$$

where $q_l(x)=\mathbb{P}_0(\eta_n\ne 0$ for every $1\le n\le l-1$ and $\eta_l=x)$. Also,

$$\begin{aligned}
\sum_{y\in\mathbb{Z}^d}\sum_{0\le s\le n}\mathbb{P}_0(T_y^{(1)}=s)q_{n-s}(x-y)&=\sum_{y\in\mathbb{Z}^d}\mathbb{P}_0(T_y^{(1)}\le n,T_y^{(2)}>n,\eta_n=x)\\
&=\mathbb{E}_0[R_n(1)I(\eta_n=x)]\\
&\le n\mathbb{P}_0(\eta_n=x).
\end{aligned}$$

Therefore, we have

$$\mathbb{E}_0[R_n(j)I(\eta_n=x)]\le\sum_{0\le t\le n}(n-t)\mathbb{P}_0(T_0^{(j-1)}=t)\mathbb{P}_0(\eta_{n-t}=x). \tag{3.5}$$

Now, set $k=[\alpha|x|]$. We show that there exists a constant $C>0$ such that for every $\alpha>0$ large enough there exists $r_0=r_0(\alpha)>0$ and it holds that $\mathbb{P}_0(\eta_l=x)\le C\mathbb{P}_0(\eta_k=x)$ for every x with $|x|\ge r_0$ and $l\le k=[\alpha|x|]$. At first, take $\alpha>\frac{1}{a_0}$ large enough where a_0 is a constant in Lemma 3.2 and consider the case that $\frac{1}{a_0\alpha}<\frac{l}{k}\le 1$. By applying Lemma 3.2 for $\mathbb{P}_0(\eta_l=x)$ and $\mathbb{P}_0(\eta_k=x)$, we have

$$\begin{aligned}
\frac{\mathbb{P}_0(\eta_l=x)}{\mathbb{P}_0(\eta_k=x)}&\le C\exp\left\{-lI\left(\frac{x}{l}\right)+kI\left(\frac{x}{k}\right)\right\}\\
&\le C.
\end{aligned}$$

Note that $\frac{1}{a}I(ax) \le \frac{1}{b}I(bx)$ for every $0 < a \le b$ and x since I is a convex function and $I(0) = 0$ (cf. the proof of Proposition B.2 of [4]). The case of small $\frac{l}{k}$ easily follows from LDP estimate for $\mathbb{P}_0(\eta_l = x)$ and Lemma 3.2 for $\mathbb{P}_0(\eta_k = x)$. Hence we obtain

$$\begin{aligned}\mathbb{E}_0[R_k(j)I(\eta_k = x)] &\le C \sum_{0\le t\le k} k\mathbb{P}_0(T_0^{(j-1)} = t)\mathbb{P}_0(\eta_k = x)\\ &\le Ck\mathbb{P}_0(T_0^{(j-1)} < \infty)\mathbb{P}_0(\eta_k = x)\\ &= Ck(1-\gamma)^{j-1}\mathbb{P}_0(\eta_k = x).\end{aligned}$$

Also, by (3.5)

$$\begin{aligned}\sum_{n\ge 1}\mathbb{E}_0[R_n(j)I(\eta_n = x)] &\le \sum_{n\ge 1}\sum_{0\le t\le n}(n-t)\mathbb{P}_0(T_0^{(j-1)} = t)\mathbb{P}_0(\eta_{n-t} = x)\\ &= \sum_{t\ge 1}\left\{\sum_{n\ge t+1}\mathbb{P}_0(T_0^{(j-1)} = n-t)\right\}t\mathbb{P}_0(\eta_t = x)\\ &\le (1-\gamma)^{j-1}\sum_{t\ge 1}t\mathbb{P}_0(\eta_t = x).\end{aligned}$$

By local central limit theorem, $\sum_{t\ge 1} t\mathbb{P}_0(\eta_t = x) \le C < \infty$ if $d \ge 5$. □

4 Variance Estimate

In this section, we prove the variance estimates (1.6) and (1.7).

Proof of (1.6) Equation (1.6) easily follows from the proof of (1.4). By (2.1), we have

$$\begin{aligned}E^{P_{N,m}}[(\phi_0)^2|\mathcal{W}_{T_N}(L)] &\le {\sum_{\omega:0\to 0}}'\left(\frac{1}{2d(m^2+1)}\right)^{|\omega|}(q_{N,m}^L)^{|R(\omega)|}\\ &\le q_{N,m}^L{\sum_{\omega:0\to 0}}'\left(\frac{1}{2d(m^2+1)}\right)^{|\omega|}.\end{aligned}$$

Also by random walk representation,

$$E^{P_{N,m}}[(\phi_0)^2] = {\sum_{\omega:0\to 0}}'\left(\frac{1}{2d(m^2+1)}\right)^{|\omega|}.$$

Therefore,

$$E^{P_{N,m}}[(\phi_0)^2|\mathcal{W}_{T_N}(L)] - E^{P_{N,m}}[(\phi_0)^2] \le (q_{N,m}^L - 1){\sum_{\omega:0\to 0}}'\left(\frac{1}{2d(m^2+1)}\right)^{|\omega|}.$$

By taking the limit $N \to \infty$ and Lemma 2.2, we have

$$E^{P_{\infty,m}^L}[(\phi_0)^2] - G_m \le -Ce^{-(\frac{1}{2G_m}+\delta)L^2},$$

for every L large enough. Finally by taking the limit $m \to 0$, we obtain (1.6). □

Proof of (1.7) At first, by Lemma 2.1,

$$E^{P_{N,m}}[(\phi_0)^2] - E^{P_{N,m}}[(\phi_0)^2|\mathcal{W}_{T_N}(L)]$$

$$= \sum_{\omega:0\to 0}{}' \left(\frac{1}{2d(m^2+1)}\right)^{|\omega|} \int \frac{\Xi^L_{N,m} - \Xi^{(}_{N,m}\frac{2}{m^2+1}\psi)}{\Xi^L_{N,m}} \mu_\omega(d\psi)$$

$$= \sum_{\omega:0\to 0}{}' \left(\frac{1}{2d(m^2+1)}\right)^{|\omega|}$$

$$\times \int P_{N,m}\left(\phi_z^2 + \frac{2}{m^2+1}\psi_z > L^2 \text{ for some } z \in T_N \;\Big|\; \mathcal{W}_{T_N}(L)\right)\mu_\omega(d\psi).$$

We estimate that

$$\widetilde{I}_\omega \equiv \int P_{N,m}\left(\phi_z^2 + \frac{2}{m^2+1}\psi_z > L^2 \text{ for some } z \in T_N \;\Big|\; \mathcal{W}_{T_N}(L)\right)\mu_\omega(d\psi)$$

$$= \int P_{N,m}\left(\phi_z^2 + \frac{2}{m^2+1}\psi_z > L^2 \text{ for some } z \in R(\omega) \;\Big|\; \mathcal{W}_{T_N}(L)\right)$$

$$\times \prod_{z\in R(\omega)} \mu_{n(z,\omega)}(d\psi_z)$$

$$\le \sum_{z\in R(\omega)} \int P_{N,m}\left(\phi_z^2 + \frac{2}{m^2+1}\psi_z > L^2 \;\Big|\; \mathcal{W}_{T_N}(L)\right)\mu_{n(z,\omega)}(d\psi_z)$$

$$\le \sum_{z\in R(\omega)} P_{N,m}\otimes\mu_{n(z,\omega)}\left(\phi_z^2 + \frac{2}{m^2+1}\psi_z > L^2\right),$$

where the last inequality follows from Griffith's inequality. Combining these estimates, we obtain

$$E^{P_{N,m}}[(\phi_0)^2] - E^{P_{N,m}}[(\phi_0)^2|\mathcal{W}_{T_N}(L)]$$

$$\le \sum_{\omega:0\to 0}{}' \left(\frac{1}{2d(m^2+1)}\right)^{|\omega|} \sum_{z\in R(\omega)} P_{N,m}\otimes\mu_{n(z,\omega)}\left(\phi_z^2 + \frac{2}{m^2+1}\psi_z > L^2\right)$$

$$= \sum_{k=1}^{\infty} \left(\frac{1}{2d(m^2+1)}\right)^{k} \sum_{\substack{\omega:0\to 0\\ |\omega|=k}}{}' \sum_{z\in R(\omega)} P_{N,m}\otimes\mu_{n(z,\omega)}\left(\phi_z^2 + \frac{2}{m^2+1}\psi_z > L^2\right).$$

Taking the limit $N \to \infty$,

$$E^{P_{\infty,m}}[(\phi_0)^2] - E^{P^L_{\infty,m}}[(\phi_0)^2]$$

$$\le \sum_{k=1}^{\infty} \left(\frac{1}{m^2+1}\right)^{k} \mathbb{E}_0\Bigg[\sum_{z\in\eta_{[0,k]}} \widetilde{q}^L_m(n(z,\eta_{[0,k]}))I(\eta_k=0)\Bigg]$$

$$\leq \mathbb{E}_0\left[\sum_{k=1}^{\infty}\sum_{j=1}^{k} R_k(j)\widetilde{q}_m^L(j) I(\eta_k=0)\right]$$
$$=\sum_{j=1}^{\infty}\left\{\sum_{k=j}^{\infty}\mathbb{E}_0[R_k(j)I(\eta_k=0)]\right\}\widetilde{q}_m^L(j),$$

where $\widetilde{q}_m^L(j) = P_{\infty,m}\otimes\mu_j(\phi_0^2+\frac{2}{m^2+1}\psi_0 > L^2)$, $j\in\mathbb{N}$. By using Lemma 3.1 we have

$$E^{P_{\infty,m}}[(\phi_0)^2] - E^{P^L_{\infty,m}}[(\phi_0)^2] \leq \sum_{j=1}^{\infty} C(1-\gamma)^{j-1}\widetilde{q}_m^L(j),$$

and by the proof of (3.2), we know that

$$\widetilde{q}_m^L(j) \leq \left\{p(L)e^{-\frac{1}{2G_m}L^2}\left(\frac{1}{1-\theta_m}\right)^j\right\}\wedge 1,$$

where $\theta_m = \frac{1}{(m^2+1)G_m}$. The conclusion follows from the similar argument to the proof of (1.5). □

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